NON-ABELIAN HIGHER DERIVED BRACKETS

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ABSTRACT. Let M be a graded Lie algebra that splits, as a graded space, in the direct sum of graded Lie subalgebras L and A, with A abelian. Let D be a degree one derivation of M such that $D^2 = 0$ and $D(L) \subset L$, then Voronov's construction of higher derived brackets associates to D a L_∞-structure on $A[-1]$. It has already been observed, and it follows from the results of this paper, that the resulting L_∞-algebra is a weak model for the homotopy fiber of the inclusion of differential graded Lie algebras $i : (L, D, [\cdot,\cdot]) \to (M, D, [\cdot,\cdot])$. We prove this result using standard homology perturbation theory, in this way we also extend the construction when the assumption A abelian is dropped: the resulting formulas involve Bernoulli numbers. Finally we apply the developed theory to recover a number of results scattered in the literature by Bering, Cattaneo and Schätz, Chuang and Lazarev, Getzler.

Notations and conventions

We work over a field $\mathbb{K}$ of characteristic zero, graded will mean $\mathbb{Z}$-graded. We use the notations and conventions on graded spaces set in [5], except we will denote with $s^{-1}V := V[1]$ the desuspension of a graded space V. Let $V = \oplus_{i\in\mathbb{Z}} V^i$ be a graded space, we will denote with $SV = \oplus_{i\geq 0} V^{\odot i}$ (resp.: $\overline{SV} = \oplus_{i\geq 1} V^{\odot i}$) the non reduced (resp.: reduced) symmetric coalgebra over V (where $V^{\odot n}$ is the space of *coinvariants* of $V^{\otimes n}$ under the natural action of the symmetric group S_n, with the usual Koszul rule for twisting signs, $V^{\odot 0} = \mathbb{K}$), and with $\odot$ the symmetric tensor product.

For a graded Lie algebra M and a graded Lie subalgebra $L \subset M$ we denote with $\mathrm{Der}(M)$ the graded Lie algebra of derivations of M and with $\mathrm{Der}(M, L) \subset \mathrm{Der}(M)$ the graded Lie subalgebra of derivations D such that $D(L) \subset L$.

1. Introduction

Higher derived brackets were introduced in a series of papers [15, 16] by Th. Voronov, as a mean to produce homotopy Lie algebras from simple input data (other constructions had been previously been considered by Koszul [10] and Kosmann-Schwarzbach [?]). Let M be a graded Lie algebra that splits, as a graded space, in the direct sum $M = L \oplus A$, with L and A graded Lie subalgebras and A abelian. For every element m of M the higher derived brackets on A induced by m were defined in [15] as the graded symmetric maps

$$\Phi(m)_i : A^{\odot i} \to A : a_1 \odot \cdots \odot a_i \to P[\cdots[[m, a_1], a_2]\cdots, a_i]$$

for $i \geq 1$, $\Phi(m)_0 = Pm \in A = \mathrm{Hom}(A^{\odot 0}, A)$, where $P : M \to A$ is the projector with kernel L and graded symmetry follows easily from the hypothesis that A is abelian. The main result of [15] implies that if m is a Maurer Cartan element of M, that is $m \in M^1$ is degree one and $[m, m] = 0$, then the $\Phi(m)_i$ are the structure maps (Taylor coefficients) of a curved $L_\infty[1]$-structure on A. Recall that this means that if $\mathrm{Coder}(SA)$ is the graded Lie algebra of coderivations of the symmetric coalgebra SA and $p : SA \to A$ is the natural projection, then under the corestriction

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isomorphism

$$\mathrm{Coder}(SA)\xrightarrow{\cong}\mathrm{Hom}(SA,A)=\prod_{i\geq 0}\mathrm{Hom}(A^{\odot i},A):Q\longrightarrow pQ$$

the (degree one) coderivation $\Phi(m)=(\Phi(m)_0,\ldots,\Phi(m)_i,\ldots)$ squares to zero. In particular when $m\in L$ we have $\Phi(m)_0=0$, thus $\Phi(m)$ restricts to a coderivation of the reduced symmetric coalgebra $\overline{SA}$: in this case under the décalage isomorphisms $\mathrm{Hom}(A^{\odot i},A)\xrightarrow{\cong}\mathrm{Hom}(A[-1]^{\wedge i},A[-1])[1-i]$ (cf. [5]) the $\Phi(m)_i$ turn into the structure maps of a strong homotopy Lie algebra structure, or L_∞-structure, on $A[-1]$, as defined for the first time by Lada and Stasheff in [12].

In [16] the construction was extended to arbitrary derivations D of M: in this case the higher derived brackets on A induced by D are defined to be the graded symmetric maps

$$\Phi(D)_i:A^{\odot i}\to A:a_1\odot\cdots\odot a_i\to P[\cdots[Da_1,a_2]\cdots,a_n]$$

for $i\geq 1$, then the results of [16] imply that if D is degree one, $D^2=0$ and moreover $D(L)\subset L$, then under décalage the $\Phi(D)_i$ are the Taylor coefficients of an ordinary L_∞-structure on $A[-1]$. In fact a more general statement was proved for both cases in [16], Theorem 3: that the correspondences

$$\Phi:M\to\mathrm{Coder}(SA):m\to\Phi(m)=(\Phi(m)_0,\ldots,\Phi(m)_i,\ldots)$$

$$\Phi:\mathrm{Der}(M,L)\to\mathrm{Coder}(\overline{SA}):D\to\Phi(D)=(\Phi(D)_1,\ldots,\Phi(D)_i,\ldots)$$

are morphisms of graded Lie algebras.

In the set up of the previous paragraph, it was also observed in [16], Corollary 4.1, that $A[-1]$ with the L_∞-structure given by décalage of $\Phi(D)$ is a weak model, in the homotopy category of L_∞ - algebras and morphisms, for the homotopy fiber of the inclusion of differential graded Lie algebras $i:(L,D,[\cdot,\cdot])\to(M,D,[\cdot,\cdot])$. Heuristically this explains the applicability of the higher derived brackets construction to the study of many problems in deformation theory ([6, 14]), in fact homotopy fibers naturally occur when considering semitrivial deformation problems (cf. [13, 5]): consider for instance the case of deformations of a coisotropic submanifold of a Poisson manifold, where the usual approach using higher derived brackets [14] can be paralleled with the more recent one in [1] using homotopy fibers, cf. Example 5.6. Finally it was suggested at the very end of [16] that this relation with homotopy theory could be exploited to study higher derived brackets when the abelianity assumption on A is dropped: this is what we try to do in the present paper.

We consider a L_∞ model for the homotopy fiber of $i:(L,D,[\cdot,\cdot])\to(M,D,[\cdot,\cdot])$ introduced by Fiorenza and Manetti in [5]: the tangent complex is the usual mapping cocone of the inclusion $i:(L,D)\to(M,D)$ in the category of complexes, and $A[-1]$ with the differential given by décalage of $\Phi(D)_1=PD$ is a homotopy retract of it; moreover, this remains true with no assumptions on A needed except to be an algebraic complementary to L in M. From standard homotopical transfer of structure (cf. [5, 9]) it is induced a L_∞-structure on $A[-1]$: this can be computed explicitly using well know formulas (cf. loc. cit.), and in fact when A is an abelian Lie subalgebra of M one recovers this way the L_∞-structure given by décalage of $\Phi(D)$. The computation can be carried over for A a graded Lie subalgebra of M, not necessarily abelian, though we will not present it: this is how we originally arrived at the following definition.

Definition 1.1. Let M be a graded Lie algebra that splits, as a graded space, in the direct sum $M=L\oplus A$ of graded Lie subalgebras L and A. For every $m\in M$ we define the higher derived brackets $\Phi(m)_i:A^{\odot i}\to A$ on A induced by m, for $i\geq 0$, by $\Phi(m)_0=Pm\in A=\mathrm{Hom}(A^{\odot 0},A)$ and for $i\geq 1$

$$\Phi(m)_i(a_1\odot\cdots\odot a_i)=\sum_{\sigma\in S_i}\varepsilon(\sigma)\sum_{k=0}^{i}\frac{B_{i-k}}{k!(i-k)!}\overbrace{[\cdots[}^{i-k}P([\cdots[m,a_{\sigma(1)}]\cdots,a_{\sigma(k)}]),a_{\sigma(k+1)}]\cdots,a_{\sigma(i)}]$$

where $\varepsilon(\sigma)$ is the Koszul sign of the permutation σ applied to $a_1\otimes\cdots\otimes a_i$ and the B_j are the Bernoulli numbers.

For every $D\in\mathrm{Der}(M,L)$ we define the higher derived brackets $\Phi(D)_i: A^{\odot i}\to A$ on A induced by D, for $i\geq 1$, by

$$\Phi(D)_i(a_1\odot\cdots\odot a_i)=\sum_{\sigma\in S_i}\varepsilon(\sigma)\sum_{k=1}^{i}\frac{B_{i-k}}{k!(i-k)!}\overbrace{[\cdots[}^{i-k}P([\cdots[Da_{\sigma(1)},a_{\sigma(2)}]\cdots,a_{\sigma(k)}]),a_{\sigma(k+1)}]\cdots,a_{\sigma(i)}]$$

For instance

$$\Phi(D)_1(a)=PDa,\qquad \Phi(m)_1(a)=P[m,a]-\frac{1}{2}[Pm,a],$$
$$\Phi(D)_2(a_1\odot a_2)=\sum_{\sigma\in S_2}\varepsilon(\sigma)\left(\frac{1}{2}P[Da_{\sigma(1)},a_{\sigma(2)}]-\frac{1}{2}[PDa_{\sigma(1)},a_{\sigma(2)}]\right),$$
$$\Phi(m)_2(a_1\odot a_2)=\sum_{\sigma\in S_2}\varepsilon(\sigma)\left(\frac{1}{2}P[[m,a_{\sigma(1)}],a_{\sigma(2)}]-\frac{1}{2}[P[m,a_{\sigma(1)}],a_{\sigma(2)}]+\frac{1}{12}[[Pm,a_{\sigma(1)}],a_{\sigma(2)}]\right).$$

Some remarks are in order with the above definition: first, the brackets take values in A since A is supposed to be $[\cdot,\cdot]$-closed; second, even for $i\geq 1$ and $\mathrm{ad}\ m\in\mathrm{Der}(M,L)$ (that is, m in the normalizer of L in M) we have $\Phi(m)_i\neq\Phi(\mathrm{ad}\ m)_i$ unless $m\in L$, so the two constructions shouldn't be confused in general (for $m\in L$, however, we have $\mathrm{ad}\ m\in\mathrm{Der}(M,L)$ and $\Phi(m)=\Phi(\mathrm{ad}\ m)$); finally, the occurrence of Bernoulli numbers in the formulas for the brackets shouldn't be too surprising, in light of their occurrence already in the formulas from [5], as well as in some related computations from Bering [2] and Getzler [7]. That this is a reasonable definition follows from the fact that they reduce to the ones introduced by Voronov when A is abelian (this is straightforward, cf. Lemma 4.3), from the sketched connection with homotopy theory and from the following theorem, which will be proved in Section 4.

Theorem 1.2. *If $M=L\oplus A$, L and A graded Lie subalgebras, then the following set of identities hold for every $D,D_k\in\mathrm{Der}(M,L)$, $m,m_k\in M$, $k=1,2$:*

$$[\Phi(m_1),\Phi(m_2)]=\Phi([m_1,m_2]) \tag{1.1}$$

$$[\Phi(D_1),\Phi(D_2)]=\Phi([D_1,D_2]) \tag{1.2}$$

$$[\Phi(D),\Phi(m)]=\Phi(Dm) \tag{1.3}$$

where the bracket in the LHS is the usual (Nijenhuis-Richardson) bracket of coderivations. Thus, in particular, the higher derived brackets constructions define morphisms of graded Lie algebras $\Phi: M\to\mathrm{Coder}(SA): m\to\Phi(m)$ and $\Phi:\mathrm{Der}(M,L)\to\mathrm{Coder}(\overline{SA}): D\to\Phi(D)$.

To prove the above theorem we imitate a construction from [16], Section 4: namely we show that if $D\in\mathrm{Der}^1(M,L)$ and $D^2=0$ we can put a $L_\infty[1]$-structure $R=(r_1,\ldots,r_i,\ldots)$ on $s^{-1}M\oplus A$, with the family of Taylor coefficients $r_i:(s^{-1}M\oplus A)^{\odot i}\to s^{-1}M\oplus A$ defined, under the isomorphism $(s^{-1}M\oplus A)^{\odot i}=\oplus_{k=0}^{i}(s^{-1}M)^{\odot k}\otimes A^{\odot i-k}$, by

$$r_1(s^{-1}m,a)=(-s^{-1}Dm,P(Da+m)),$$
$$r_2(s^{-1}m_1\odot s^{-1}m_2)=(-1)^{|m_1|}s^{-1}[m_1,m_2],$$
$$r_i(a_1\odot\cdots\odot a_i)=\Phi(D)_i(a_1\odot\cdots\odot a_i),$$
$$r_{i+1}(s^{-1}m\otimes a_1\odot\cdots\odot a_i)=\Phi(m)_i(a_1\odot\cdots\odot a_i),$$

for $i \geq 1$, and $R = 0$ otherwise. The corresponding L_∞-algebra is a mapping cocylinder, in the category of L_∞-algebras and morphisms, of the inclusion of differential graded Lie algebras $i : (L, D, [\cdot,\cdot]) \to (M, D, [\cdot,\cdot])$: in particular the construction implies that $A[-1]$ with the L_∞-structure given by décalage of $\Phi(D)$ is a homotopy fiber of i.

The last section is devoted to applications of the developed theory. We are able to easily recover a number of results scattered in the literature, these are: results of Cattaneo and Schätz [3] about some invariance properties of the higher derived brackets construction; a theorem of Chuang and Lazarev [4], stating that every $L_\infty[1]$-algebra (V, Q), $Q \in \mathrm{Coder}^1(\overline{SV})$, $[Q, Q] = 0$, is homotopy equivalent to the Quillen construction on the homotopy fiber of the inclusion of DGLAs

$$i : (\mathrm{Coder}(\overline{SV}), [Q,\cdot], [\cdot,\cdot]) \to (\mathrm{Coder}(SV), [Q,\cdot], [\cdot,\cdot]);$$

a construction of higher derived brackets on the negatively graded part of any DGLA due to Getzler [7]; (conjecturally) a hierarchy of higher brackets, introduced by Bering [2], associated to an operator over a unital associative algebra A, reducing to the usual higher Koszul brackets from [10] when A is commutative; the extension of the results of the author and M. Manetti [1] to the study of coisotropic deformations in the differentiable setting. Finally we obtain a simple necessary and sufficient condition for a L_∞-algebra to be homotopy abelian (Theorem 5.4) of which the author, most likely due to his own ignorance, wasn't aware.

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2. Review of $L_\infty[1]$-algebras

In the body of the paper we will work, mostly, in the category of $L_\infty[1]$-algebras and morphisms. This is isomorphic to the usual category of L_∞-algebras from [12] via the so called décalage isomorphisms (cf. [5] for the definition): these give a bijective correspondence from the set of L_∞-structures on a graded space V to the set of $L_\infty[1]$-structures on the space $s^{-1}V = V[1]$.

Let $p : SV \to V^{\odot 1} = V$ (resp.: $p : \overline{SV} \to V$) denote the natural projection, then corestriction induces isomorphisms of graded spaces from the the graded Lie algebras $\mathrm{Coder}(SV)$ (resp.: $\mathrm{Coder}(\overline{SV})$) of coderivations

$$\mathrm{Coder}(SV) \xrightarrow{\cong} \mathrm{Hom}(SV, V) = \prod_{i\geq 0} \mathrm{Hom}(V^{\odot i}, V) : Q \longrightarrow pQ$$

$$\mathrm{Coder}(\overline{SV}) \xrightarrow{\cong} \mathrm{Hom}(\overline{SV}, V) = \prod_{i\geq 1} \mathrm{Hom}(V^{\odot i}, V) : Q \longrightarrow pQ$$

If $Q \in \mathrm{Coder}(SV)$ let $pQ = q = (q_0, \ldots, q_i, \ldots)$, we call $q_i : V^{\odot i} \to V$ the i-th Taylor coefficient of Q, notice that the 0-th Taylor coefficient q_0 is just an element of $\mathrm{Hom}(V^{\odot 0}, V) = V$; similarly if $Q \in \mathrm{Coder}(\overline{SV})$. The inverse of the first isomorphism sends $(q_0, \ldots, q_i, \ldots)$ to the coderivation given by $Q(1) = q_0$ and

$$Q(v_1 \odot \cdots \odot v_i) = q_0 \odot v_1 \odot \cdots \odot v_i + \sum_{k=1}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(i)}$$

where $S(i,j)$ is the set of (i,j)-unshuffles, i.e., permutations $\sigma \in S_{i+j}$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(i+j)$, and $\varepsilon(\sigma) = \varepsilon(\sigma; v_1, \ldots, v_i)$ is the Koszul sign. The inverse of the second isomorphism is given by the above formula minus the term $q_0 \odot v_1 \odot \cdots \odot v_i$. Coderivations such that $q_i = 0$ for $i \neq 1$ are called linear.

Remark 2.1. There is a natural embedding $i : \mathrm{Coder}(\overline{SV}) \to \mathrm{Coder}(SV)$, given in Taylor coefficients by $(q_1, \dots, q_i, \dots) \to (0, q_1, \dots, q_i, \dots)$, that identifies $\mathrm{Coder}(\overline{SV})$ with the sub Lie algebra of coderivations $Q \in \mathrm{Coder}(SV)$ such that $Q(1) = 0$. This embedding fits in an exact sequence of graded spaces

$$0 \to \mathrm{Coder}(\overline{SV}) \xrightarrow{i} \mathrm{Coder}(SV) \xrightarrow{e} V \to 0 \tag{2.1}$$

where e is the evaluation morphism $Q \to Q(1)$.

On $\mathrm{Coder}(SV)$ it is defined a right pre-Lie operation, that we denote by $\bullet$, inducing the usual Lie bracket, namely $Q \bullet R$ is the coderivation given in Taylor coefficients by pQR. More explicitly:

$$(Q \bullet R)_i(v_1 \odot \cdots \odot v_i) = \sum_{k=0}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) q_{i-k+1}(r_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(i)})$$

The bracket on $\mathrm{Coder}(SV)$ can be calculated in Taylor coefficients by the above expression via the rule $[Q, R]_i = (Q \bullet R)_i - (-1)^{|Q||R|}(R \bullet Q)_i$.

Remark 2.2. Given $v \in V$, denote with $L_v \in \mathrm{Coder}(SV)$ the coderivation given in Taylor coefficients by $(v, 0, \dots, 0, \dots)$, i.e.,

$$L_v(1) = v, \qquad L_v(v_1 \odot \cdots \odot v_i) = v \odot v_1 \odot \cdots \odot v_i$$

Notice that $L : V \to \mathrm{Coder}(SV) : v \to L_v$ splits the exact sequence (2.1), and the image is an abelian Lie subalgebra of $\mathrm{Coder}(SV)$. Also notice that $[Q, L_v] = Q \bullet L_v$, for every $v \in V$ and $Q \in \mathrm{Coder}(SA)$, is the coderivation given by $[Q, L_v]_i(v_1 \odot \cdots \odot v_i) = q_{i+1}(v \odot v_1 \odot \cdots \odot v_i)$ for every $i \geq 0$.

Finally, given graded spaces V, W, every morphism of graded coalgebras $F : \overline{SV} \to \overline{SW}$ is determined by its corestriction $(f_1, \dots, f_i, \dots) = pF : \overline{SV} \to W$, by the rule

$$F_i^k(v_1 \odot \cdots \odot v_i) = \frac{1}{k!} \sum_{j_1 + \cdots + j_k = i} \sum_{\sigma \in S(j_1, \dots, j_k)} \varepsilon(\sigma) f_{j_1}(v_{\sigma(1)} \odot \cdots) \odot \cdots \odot f_{j_k}(\cdots \odot v_{\sigma(i)})$$

where F_i^k is the composition $V^{\odot i} \to \overline{SV} \xrightarrow{F} \overline{SW} \to W^{\odot k}$, and $F_i^k = 0$ if $k > i$ (and conversely this defines a morphism of coalgebras for any $(f_1 \dots, f_i, \dots) : \overline{SV} \to W$). Recall ([9]) that F is an isomorphism (resp.: monomorphism, epimorphism) if and only if its linear part f_1 is. A morphism F such that $f_i = 0$ for $i \geq 2$ is called linear.

Definition 2.3. Given a graded space V, a $L_\infty[1]$-algebra structure on V is the datum of a degree one $(q_1, \dots, q_i, \dots) = Q \in \mathrm{Coder}^1(\overline{SV})$ such that $Q^2 = Q \bullet Q = \frac{1}{2}[Q, Q] = 0$. Given $L_\infty[1]$-algebras (V, Q) and (W, R), a $L_\infty[1]$-morphism between them is a coalgebra morphism $F : \overline{SV} \to \overline{SW}$ commuting with the codifferentials, i.e., such that $FQ - RF = 0$. A $L_\infty[1]$-algebra (V, Q) is called abelian if Q is a linear coderivation.

Remark 2.4. In particular $0 = {q_1}^2$, the differential graded (DG) space (V, q_1) is called the tangent complex of the $L_\infty[1]$-algebra (V, Q), as in remark 2.2 it fits in an exact sequence of DG spaces

$$0 \to (\mathrm{Coder}(\overline{SV}), [Q, \cdot]) \xrightarrow{i} (\mathrm{Coder}(SV), [Q, \cdot]) \xrightarrow{e} (V, q_1) \to 0 \tag{2.2}$$

Example 2.5. Given a differential graded Lie algebra (DGLA) $(L, d, [\cdot, \cdot])$, it is defined a $L_\infty[1]$-structure Q on $s^{-1}L$ by the rule $q_1(s^{-1}l) = -s^{-1}dl$, $q_2(s^{-1}l_1 \odot s^{-1}l_2) = (-1)^{|l_1|}s^{-1}[l_1, l_2]$, $q_i = 0$ if $i \geq 3$, this is called the Quillen construction on L ([8]). A morphism of DGLAs induces a linear morphism between the corresponding Quillen constructions, in this way the category of DGLAs is faithfully (not fully) embedded in the category of $L_\infty[1]$-algebras.

A morphism $F:(V,Q)\to(W,R)$ of $L_\infty[1]$-algebras is a homotopy equivalence (or weak equivalence) if its linear part f_1 is a quasi isomorphism of the tangent complexes.

Definition 2.6. A $L_\infty[1]$-algebra (V,Q) is homotopy abelian if it is homotopy equivalent to an abelian one. Consider q_1 as an abelian $L_\infty[1]$-structure on V: structure theory of $L_\infty[1]$-algebras (cf. [9]) shows that (V,Q) is homotopy abelian if and only if (V,Q) and (V,q_1) are isomorphic $L_\infty[1]$-algebras.

We end this section by recalling the theorem on homotopical transfer of structure: this says that $L_\infty[1]$-structures (unlike, for instance, DGLA structures) can be transferred along homotopy retractions. It is a major result in the theory of $L_\infty[1]$-algebras, since it allows to define a (well defined up to isomorphism) $L_\infty[1]$-structure on the homology of the tangent complex of a given $L_\infty[1]$-algebra (the resulting $L_\infty[1]$-algebra is known as the minimal model) retaining all of the homotopical information about the original algebra, cf. [9]. The version we give here is taken from [5], Theorem 4.1, for a nice proof the reader is referred to the arXiv version of the paper.

Definition 2.7. Given a pair of DG spaces (V,q_1) and (W,r_1), we call homotopy retraction data from V to W the data of a pair of DG morphisms $\pi:V\to W$, $f_1:W\to V$ and a contracting homotopy $K\in\operatorname{Hom}^{-1}(V,V)$ such that

$$\pi f_1=\mathrm{id}_W$$

$$Kq_1+q_1K=f_1\pi-\mathrm{id}_V$$

Theorem 2.8. *(Homotopical transfer of structure) Let (V,Q) be a $L_\infty[1]$-algebra, (W,r_1) a DG space and π, f_1, K homotopy retraction data from (V,q_1) to (W,r_1) as in the previous definition. Let $q=pQ$, $q_+=q-q_1$, then there exists an unique morphism of coalgebras $F:\overline{SW}\to\overline{SV}$ such that $pF=f_1+Kq_+F$. Moreover, if $R\in\operatorname{Coder}^1(\overline{SW})$ is the unique coderivation with corestriction $pR=r_1+\pi q_+F$, then R is a $L_\infty[1]$-structure on W and $F:(W,R)\to(V,Q)$ a $L_\infty[1]$-morphism.*

Notice that homotopical transfer of structure produces a weak equivalent $L_\infty[1]$-algebra, with $F:(W,R)\to(V,Q)$ an explicit homotopy equivalence.

3. Mapping cocone and mapping cocylinder

Given a DGLA $(L,d_L,[\cdot,\cdot])$ and a differential graded algebra (DGA) $(A,d_A,\cdot)$ there is a natural DGLA structure on the tensor product $A\otimes L$, with differential $d_A\otimes\mathrm{id}_L+\mathrm{id}_A\otimes d_L$ and bracket $[a_1\otimes l_1,a_2\otimes l_2]=(-1)^{|l_1||a_2|}a_1a_2\otimes[l_1,l_2]$. When $A=\mathbb{K}[t,dt]$ is the DGA of polynomial forms on the line (that is the graded $\mathbb{K}$-algebra freely generated by t in degree zero and dt in degree one, and with differential $d(t)=dt$ and $d(dt)=0$) and L a DGLA the corresponding DGLA will be denoted with $L[t,dt]$. For every $s\in\mathbb{K}$ the evaluation $e_s:L[t,dt]\xrightarrow{t=s,dt=0}L$ is a quasi isomorphism, left inverse to the natural inclusion $L\to L[t,dt]:l\to 1\otimes l$; formal integration in t induces degree minus one operators $\int_0^t:L[t,dt]\to L[t,dt]$ and $\int_0^1:L[t,dt]\to L$.

Definition 3.1. Given a pair of morphisms of DGLAs $f:L\to M$ and $g:N\to M$, the homotopy fiber product of L and N along f and g is by definition the DGLA

$$L\times^h_M N=\{(l,n,m(t,dt))\in L\times N\times M[t,dt]\text{ s.t. }e_0(m(t,dt))=f(l),\ e_1(m(t,dt))=g(n)\}$$

Given a DGLA morphism $f:L\to M$ the homotopy fiber of f is by definition the homotopy fiber product $K_f=0\times^h_M L$ along the trivial morphism and f.

Denote with $\{B_i\}_{i\in\mathbb{N}}$ the sequence of Bernoulli numbers, i.e., the sequence of (rational) numbers defined by the power series expansion at zero of $\frac{t}{e^t-1}=\sum_{i\geq 0}\frac{B_i}{i!}t^i = 1-\frac{1}{2}t+\frac{1}{2!}\frac{1}{6}t^2+\frac{1}{4!}(-\frac{1}{30})t^4+\cdots$. This is sometimes called the first sequence of Bernoulli numbers, occasionally we will use the notation $B_i(0):=B_i$ to distinguish it from the second sequence of Bernoulli numbers defined by $B_i(1):=(-1)^iB_i$; recall that $B_{2i+1}(0)=B_{2i+1}(1)=0$ for $i\geq 1$.

The next theorem was proved in [5], Theorem 5.5.

Theorem 3.2. *Let K_f be the homotopy fiber of a morphism of DGLAs $f: L\to M$, $(s^{-1}K_f, Q)$ the $L_\infty[1]$-algebra given by the Quillen construction. Let (C_f, r_1) be the desuspended mapping cocone of f in the category of complexes, that is*

$$\mathrm{C}_f = s^{-1}L\oplus M,\qquad r_1(s^{-1}l, m)=(-s^{-1}d_Ll, d_Mm-f(l)).$$

It is defined homotopy retraction data from $(s^{-1}K_f, q_1)$ to (C_f, r_1), as in definition 2.7, with the DG morphisms $\pi(s^{-1}(l, m(t,dt)))=(s^{-1}l, \int_0^1 m(t,dt))$ and $f_1(s^{-1}l,m)=s^{-1}(l, t\cdot f(l)+dt\cdot m)$ and the contracting homotopy $K(s^{-1}(l,m(t,dt)))=s^{-1}(0,\int_0^t m(t,dt)-t\cdot\int_0^1 m(t,dt))$. The resulting $L_\infty[1]$-structure R on C_f, obtained via homotopical transfer of structure (Theorem 2.8) is given in Taylor coefficients (under the isomorphism $(s^{-1}L\oplus M)^{\odot i}=\sum_{k=0}^n (s^{-1}L)^{\odot k}\otimes M^{\odot i-k}$) by

$$r_2(s^{-1}l_1\odot s^{-1}l_2) = (-1)^{|l_1|}s^{-1}[l_1,l_2],$$
$$r_{i+1}(s^{-1}l\otimes m_1\odot\cdots\odot m_i) = -\frac{B_i}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)[\cdots[f(l), m_{\sigma(1)}]\cdots, m_{\sigma(i)}]$$

for $i\geq 1$, and $R=0$ otherwise.

We call C_f with the $L_\infty[1]$-structure from the above theorem the mapping cocone of f, we will also use a similar construction for a mapping cocylinder. Let $H_f = M\times^h_M L$ be the homotopy fiber product along id_M and f, $(s^{-1}H_f, Q)$ the $L_\infty[1]$-algebra given by the Quillen construction. Let (Cyl_f, r_1) be the desuspended mapping cocylinder of f in the category of complexes, i.e.,

$$\mathrm{Cyl}_f = s^{-1}L\oplus s^{-1}M\oplus M,\qquad r_1(s^{-1}l, s^{-1}m, n) = (-s^{-1}d_Ll, -s^{-1}d_Mm, d_Mn+m-f(l)).$$

As before we can define homotopy retraction data from $(s^{-1}H_f, q_1)$ to (Cyl_f, r_1): this time we put $\pi(s^{-1}(l,m,n(t,dt)))=(s^{-1}l, s^{-1}m, \int_0^1 n(t,dt))$,

$$f_1(s^{-1}l, s^{-1}m, n) = s^{-1}(l, m, t\cdot f(l)+(1-t)\cdot m+dt\cdot n),$$

and finally $K(s^{-1}(l,m,n(t,dt))) = s^{-1}(0,0,\int_0^t n(t,dt)-t\cdot\int_0^1 n(t,dt))$. We leave to the reader to check that this is in fact homotopy retraction data as in Definition 2.7.

Theorem 3.3. *The $L_\infty[1]$-structure $R=(r_1,\ldots,r_i,\ldots)$ on Cyl_f, obtained by applying homotopical transfer along the above defined homotopy retraction data, is given by*

$$r_2(s^{-1}l_1\odot s^{-1}l_2)=(-1)^{|l_1|}s^{-1}[l_1,l_2],\qquad r_2(s^{-1}m_1\odot s^{-1}m_2)=(-1)^{|m_1|}s^{-1}[m_1,m_2],$$
$$r_{i+1}(s^{-1}l\otimes n_1\odot\cdots\odot n_i) = -\frac{B_i(0)}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)[\cdots[f(l), n_{\sigma(1)}]\cdots, n_{\sigma(i)}],$$
$$r_{i+1}(s^{-1}m\otimes n_1\odot\cdots\odot n_i) = \frac{B_i(1)}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)[\cdots[m, n_{\sigma(1)}]\cdots, n_{\sigma(i)}]$$

for $i\geq 1$, and $R=0$ otherwise.

Proof. Omitted, this can be given following closely the argument from [5], Section 5. □

We call Cyl_f with the $L_\infty[1]$-structure from the above theorem the mapping cocylinder of f, notice that the pair of linear $L_\infty[1]$-morphisms $s^{-1}L \to \mathrm{Cyl}_f : s^{-1}l \to (s^{-1}l, s^{-1}f(l), 0)$ and $\mathrm{Cyl}_f \to s^{-1}M : (s^{-1}l, s^{-1}m, n) \to s^{-1}m$ give a factorization $s^{-1}L \to \mathrm{Cyl}_f \to s^{-1}M$ of the Quillen construction on f as a weak equivalence followed by an epimorphism (i.e., fibration) of $L_\infty[1]$-algebras.

4. Non-abelian higher derived brackets

Let M be a graded Lie algebra and suppose that M splits, as a graded space, in the direct sum $M = L \oplus A$ of graded Lie subalgebras L and A, the last one not necessarily abelian. We denote with $P : M \to A$ the projection with kernel L and with $P^\perp = \mathrm{id}_M - P$. We also denote with $\mathrm{Der}(M, L)$ the Lie algebra of derivations $D \in \mathrm{Der}(M)$ such that $D(L) \subset L$: as in [16], it is easy to see that this is equivalent to the identity

$$PDP = PD \tag{4.1}$$

for D. We are going to extend to this setting Voronov's constructions of higher derived brackets from [15, 16].

Definition 4.1. For every $m \in M$, the higher derived brackets on A induced by m are the multilinear symmetric maps $\Phi(m)_i : A^{\odot i} \to A$, $i \geq 0$, defined by

$$\Phi(m)_i(a_1 \odot \cdots \odot a_i) = \sum_{\sigma \in S_i} \varepsilon(\sigma) \sum_{k=0}^{i} \frac{B_{i-k}}{k!(i-k)!} \overbrace{[\cdots[}^{i-k} P([\cdots[m, a_{\sigma(1)}]\cdots, a_{\sigma(k)}]), a_{\sigma(k+1)}]\cdots, a_{\sigma(i)}]$$

for $i \geq 1$, $\Phi(m)_0 = Pm \in A = \mathrm{Hom}(\odot^0 A, A)$.

For every $D \in \mathrm{Der}(M, L)$ the higher derived brackets on A associated to D are the multilinear symmetric maps $\Phi(D)_i : A^{\odot i} \to A$, $i \geq 1$, defined by

$$\Phi(D)_i(a_1 \odot \cdots \odot a_i) = \sum_{\sigma \in S_i} \varepsilon(\sigma) \sum_{k=1}^{i} \frac{B_{i-k}}{k!(i-k)!} \overbrace{[\cdots[}^{i-k} P([\cdots[Da_{\sigma(1)}, a_{\sigma(2)}]\cdots, a_{\sigma(k)}]), a_{\sigma(k+1)}]\cdots, a_{\sigma(i)}]$$

We denote with $(\Phi(m)_0, \ldots, \Phi(m)_i, \ldots) = \Phi(m) \in \mathrm{Coder}(SA)$ the corresponding coderivation, and similarly for $(\Phi(D)_1, \ldots, \Phi(D)_i, \ldots) = \Phi(D) \in \mathrm{Coder}(\overline{SA})$.

Remark 4.2. If $l \in L$, then $\mathrm{ad}\, l \in \mathrm{Der}(M, L)$, and in this case the two constructions coincide, i.e., we have $\Phi(\mathrm{ad}\, l) = \Phi(l)$. However, if $\mathrm{ad}\, m \in \mathrm{Der}(M, L)$, that is m is in the normalizer of L in M, but $m \notin L$, then $\Phi(\mathrm{ad}\, m) \neq \Phi(m)$, as the two differ by the terms involving Pm.

If A is abelian, only the $k = i$ term in the summation for the brackets remains. Moreover, it was proved in [16] with a simple induction that for any derivation $\Theta \in \mathrm{Der}(M)$ the maps $A^{\otimes i} \to M : a_1 \otimes \cdots \otimes a_i \to [\cdots[\Theta a_1, a_2]\cdots, a_i]$ are graded symmetric. From this, the following lemma follows straightforwardly.

Lemma 4.3. *If A is an abelian Lie subalgebra, then the brackets in Definition 4.1 reduce to $\Phi(m)_i(a_1 \odot \cdots \odot a_i) = P[\cdots[m, a_1]\cdots, a_i]$, $i \geq 1$, $\Phi(m)_0 = Pm$, for $m \in M$, and reduce to $\Phi(D)_i(a_1 \odot \cdots \odot a_i) = P[\cdots[Da_1, a_2]\cdots, a_i]$, $i \geq 1$, for $D \in \mathrm{Der}(M, L)$, i.e., they are the same as the ones introduced by Th. Voronov in [15, 16].*

The aim of this section is to prove Theorem 1.2 from the introduction. Following the strategy outlined there, this will follow from the next proposition.

Proposition 4.4. *If $D \in \mathrm{Der}^1(M, L)$, let $R = (r_1, \dots, r_i, \dots)$ be the coderivation on $s^{-1}M \oplus A$ defined in Taylor coefficients by*

$$\begin{aligned} r_1(s^{-1}m, a) &= (-s^{-1}Dm, P(Da + m)) \\ r_2(s^{-1}m_1 \odot s^{-1}m_2) &= (-1)^{|m_1|} s^{-1}[m_1, m_2] \\ r_i(a_1 \odot \cdots \odot a_i) &= \Phi(D)_i(a_1 \odot \cdots \odot a_i) \\ r_{i+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_i) &= \Phi(m)_i(a_1 \odot \cdots \odot a_i) \end{aligned}$$

for $i \geq 1$, and $R = 0$ otherwise. Then $R^2 = ({r^2}_1, \dots, {r^2}_i, \dots)$ is the coderivation defined in Taylor coefficients by

$$\begin{aligned} {r^2}_1(s^{-1}m, a) &= {r_1}^2(s^{-1}m, a) = (s^{-1}D^2m, PD^2a) \\ {r^2}_i(a_1 \odot \cdots \odot a_i) &= \Phi(D^2)_i(a_1 \odot \cdots \odot a_i) \end{aligned}$$

for $i \geq 2$ and $R^2 = 0$ otherwise. In particular if $D^2 = 0$ then R is a $L_\infty[1]$-structure. In this case the resulting $L_\infty[1]$-algebra $(s^{-1}M \oplus A, R)$ is a mapping cocylinder, in the category of $L_\infty[1]$-algebras, of the Quillen construction on the inclusion of DGLAs $i : (L, D, [\cdot,\cdot]) \to (M, D, [\cdot,\cdot])$: in particular it is homotopy equivalent to the Quillen construction on $(L, D, [\cdot,\cdot])$.

Proof. Let (Cyl_i, Q) be the mapping cocylinder for the inclusion i as defined in Theorem 3.3. Since we are not assuming $D^2 = 0$, in general $q_1(s^{-1}l, s^{-1}m, n) = (-s^{-1}Dl, -s^{-1}Dm, Dn + m - l)$ will not be a differential, and in particular Q will not be a $L_\infty[1]$-structure on Cyl_i: we claim however that $Q^2 = {q_1}^2$ is a linear coderivation, the proof will be postponed to the end of the demonstration.

Define the following "homotopy retraction data" from (Cyl_i, q_1) to $(s^{-1}M \oplus A, r_1)$

$$\pi : \mathrm{Cyl}_i \to s^{-1}M \oplus A : (s^{-1}l, s^{-1}m, n) \longrightarrow (s^{-1}m, Pn)$$

$$f_1 : s^{-1}M \oplus A \to \mathrm{Cyl}_i : (s^{-1}m, a) \longrightarrow (s^{-1}P^\perp(Da + m), s^{-1}m, a)$$

$$K : \mathrm{Cyl}_i \to \mathrm{Cyl}_i : (s^{-1}l, s^{-1}m, n) \longrightarrow (s^{-1}P^\perp n, 0, 0)$$

When $D^2 = 0$ we are in fact the set up of Definition 2.7, we leave to the reader to check (one should recall Equation (4.1)) that in general the identities

$$\pi f_1 = \mathrm{id}_{s^{-1}M\oplus A}, \qquad Kq_1 + q_1K = f_1\pi - id_{\mathrm{Cyl}_i}, \qquad r_1\pi = \pi q_1,$$

remain satisfied.

The proof proceeds as follows: first we show that R in the claim of the proposition is induced via homotopical transfer of structure (Theorem 2.8) from Q along the above defined homotopy retraction data, that is we find $F : \overline{S(s^{-1}M \oplus A)} \to \overline{S\,\mathrm{Cyl}_i}$ such that $pF = f_1 + Kq_+F$, and then show $pR = r_1 + \pi q_+ F$, as in the claim of Theorem 2.8. Having done this, in order to compute R^2, we will give a slightly modified version of the usual proof of 2.8 (cf. [5], the arXiv version, Theorem 4.1): the argument will depend on the claim $Q^2 = {q_1}^2$.

We show that F is given in Taylor coefficients by

$$f_i(a_1 \odot \cdots \odot a_i) = \left(s^{-1}\frac{1}{i!}\sum_{\sigma\in S_i} \varepsilon(\sigma) P^\perp[\cdots[Da_{\sigma(1)}, a_{\sigma(2)}]\cdots, a_{\sigma(i)}], 0, 0 \right), \qquad i \geq 2,$$

$$f_{i+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_i) = \left(s^{-1}\frac{1}{i!}\sum_{\sigma\in S_i} \varepsilon(\sigma) P^\perp[\cdots[m, a_{\sigma(1)}]\cdots, a_{\sigma(i)}], 0, 0 \right), \qquad i \geq 1,$$

and $F = 0$ otherwise. We have to check that the identity $f_i = \sum_{j=2}^{i} Kq_j F_i^j$ holds for every $i \geq 2$, recall that F_i^j is given by the formula

$$F_i^j(\cdots) = \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma\in S(k_1,\ldots,k_j)} \varepsilon(\sigma) f_{k_1}(\cdots)\odot\cdots\odot f_{k_j}(\cdots)$$

For $i = 2$ we have

$$q_2F_2^2(a_1\odot a_2) = q_2((s^{-1}P^\perp Da_1, 0, a_1)\odot(s^{-1}P^\perp Da_2, 0, a_2)) =$$
$$= q_2(s^{-1}P^\perp Da_1\odot s^{-1}P^\perp Da_2) + q_2(s^{-1}P^\perp Da_1\otimes a_2) + (-1)^{|a_1||a_2|}q_2(s^{-1}P^\perp Da_2\otimes a_1) =$$
$$= \left((-1)^{|a_1|+1}s^{-1}[P^\perp Da_1, P^\perp Da_2],\, 0,\, \frac{1}{2}\sum_{\sigma\in S_2}\varepsilon(\sigma)([Da_{\sigma(1)}, a_{\sigma(2)}] - [PDa_{\sigma(1)}, a_{\sigma(2)}])\right)$$

Similar computations show that

$$q_2F_2^2(s^{-1}m_1\odot s^{-1}m_2) = (-1)^{|m_1|}\left(s^{-1}[P^\perp m_1, P^\perp m_2], s^{-1}[m_1,m_2], 0\right)$$
$$q_2F_2^2(s^{-1}m\otimes a) = \left((-1)^{|m|}s^{-1}[P^\perp m, P^\perp Da],\, 0,\, [m,a] - \frac{1}{2}[Pm,a]\right)$$

Using the fact that A is $[\cdot,\cdot]$-closed, we see that $Kq_2F_2^2$ and $\pi q_2F_2^2$ coincide in fact with f_2 and r_2 previously defined.

Next consider $(q_+F)_i = \sum_{j=2}^{i} q_jF_i^j$ for $i \geq 3$. To simplify the computation, notice that in order to compose with K we are only interested in $p_M(q_+F)_i$, where we denote with p_M the natural projection from $\mathrm{Cyl}_i = s^{-1}L\oplus s^{-1}M\oplus M$ to M, while in order to compose with π we also need to consider $p_{s^{-1}M}(q_+F)_i$: but the last one is always zero for $i\geq 3$. To prove this point, let $(s^{-1}M, S)$ be the Quillen construction on $(M, D, [\cdot,\cdot])$, then it is plain that considering $p_{s^{-1}M}:\mathrm{Cyl}_i\to s^{-1}M$ as a linear morphism of coalgebras $p_{s^{-1}M}Q = Sp_{s^{-1}M}$, so that $p_{s^{-1}M}q_j = 0$ for $j\geq 3$ and thus $p_{s^{-1}M}(q_+F)_i = p_{s^{-1}M}q_2F_i^2 = s_2p_{s^{-1}M}F_i^2$; it follows from the definitions that $p_{s^{-1}M}F_i^2 = 0$ for $i\geq 3$ and so we are done.

To compute $p_Mq_jF_i^j$ when $2\leq j< i$, we see by looking at the formulas for the q_j that the only part in the summation for F_i^j not lying in $\operatorname{Ker} p_Mq_j$ can be rewritten as

$$\frac{1}{(j-1)!}\sum_{\sigma\in S(i-j+1,1,\ldots,1)}\varepsilon(\sigma)f_{i-j+1}(\cdots)\odot f_1(\cdots)\odot\cdots\odot f_1(\cdots)$$

Under the isomorphism $(s^{-1}M\oplus A)^{\odot i+1} = \sum_{k=0}^{i+1} s^{-1}M^{\odot k}\otimes A^{\odot i-k+1}$ this vanishes except on terms of the form $a_1\odot\cdots\odot a_i$ and $s^{-1}m\otimes a_1\odot\cdots\odot a_{i-1}$. Using symmetry of f_{i-j+1}, for $2\leq j<i$ we can write

$$p_Mq_jF_i^j(a_1\odot\cdots\odot a_i) =$$
$$= \frac{1}{(j-1)!(i-j+1)!}\sum_{\sigma\in S_i}\varepsilon(\sigma)p_Mq_j\left(f_{i-j+1}(a_{\sigma(1)}\odot\cdots\odot a_{\sigma(i-j+1)})\odot a_{\sigma(i-j+2)}\odot\cdots\odot a_{\sigma(i)}\right) =$$
$$= -\frac{B_{j-1}}{(j-1)!(i-j+1)!}\sum_{\sigma\in S_i}\varepsilon(\sigma)\overbrace{[\cdots[}^{j-1}P^\perp([\cdots[Da_{\sigma(1)}, a_{\sigma(2)}]\cdots]), a_{\sigma(i-j+2)}]\cdots, a_{\sigma(i)}]$$

In the same way, for $2\leq j\leq i$

$$p_Mq_jF_{i+1}^j(s^{-1}m\otimes a_1\odot\cdots\odot a_i) =$$
$$= -\frac{B_{j-1}}{(j-1)!(i-j+1)!}\sum_{\sigma\in S_i}\varepsilon(\sigma)\overbrace{[\cdots[}^{j-1}P^\perp([\cdots[m, a_{\sigma(1)}]\cdots]), a_{\sigma(i-j+2)}]\cdots, a_{\sigma(i)}]$$

The remaining terms to consider are

$$p_M q_i F_i^i(a_1 \odot \cdots \odot a_i) = -\frac{B_{i-1}}{(i-1)!} \sum_{\sigma \in S_i} \varepsilon(\sigma)[\cdots[P^\perp D a_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(i)}]$$

and

$$p_M q_{i+1} F_{i+1}^{i+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_i) = p_M q_{i+1}(s^{-1}P^\perp m \otimes a_1 \odot \cdots \odot a_i) + p_M q_{i+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_i) =$$

$$= \frac{B_i}{i!} \sum_{\sigma \in S_i} \varepsilon(\sigma)[\cdots[m - P^\perp m, a_{\sigma(1)}] \cdots, a_{\sigma(i)}] = \frac{B_i}{i!} \sum_{\sigma \in S_i} \varepsilon(\sigma)[\cdots[Pm, a_{\sigma(1)}] \cdots, a_{\sigma(i)}]$$

Finally, after a change of variable $k = i - j + 1$, we see that

$$p_M q_+ F(a_1 \odot \cdots \odot a_i) = \sum_{j=2}^{i} p_M q_j F_i^j(a_1 \odot \cdots \odot a_i) =$$

$$= \sum_{k=1}^{i-1} \frac{B_{i-k}}{k!(i-k)!} \sum_{\sigma \in S_i} \varepsilon(\sigma) \overbrace{[\cdots[}^{i-k} P([\cdots[Da_{\sigma(1)}, a_{\sigma(2)}] \cdots]), a_{\sigma(k+1)}] \cdots, a_{\sigma(i)}] +$$

$$+ \left(-\sum_{k=1}^{i-1} \frac{B_{i-k}}{k!(i-k)!} \right) \sum_{\sigma \in S_i} \varepsilon(\sigma)[\cdots[Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(i)}]$$

We use the well known identity $\sum_{k=0}^{i-1} B_k \begin{pmatrix} i \\ k \end{pmatrix} = 0$ for $i \geq 2$ in order to conclude

$$p_M q_+ F(a_1 \odot \cdots \odot a_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \varepsilon(\sigma)[\cdots[Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(i)}] +$$

$$+ \sum_{k=1}^{i-1}) \frac{B_{i-k}}{k!(i-k)!} \sum_{\sigma \in S_i} \varepsilon(\sigma) \overbrace{[\cdots[}^{i-k} P([\cdots[Da_{\sigma(1)}, a_{\sigma(2)}] \cdots]), a_{\sigma(k+1)}] \cdots, a_{\sigma(i)}]$$

In the same way

$$p_M q_+ F(s^{-1}m \otimes a_1 \odot \cdots \odot a_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \varepsilon(\sigma)[\cdots[m, a_{\sigma(1)}] \cdots, a_{\sigma(i)}] +$$

$$+ \sum_{k=0}^{i-1} \frac{B_{i-k}}{k!(i-k)!} \sum_{\sigma \in S_i} \varepsilon(\sigma) \overbrace{[\cdots[}^{i-k} P([\cdots[m, a_{\sigma(1)}] \cdots]), a_{\sigma(k+1)}] \cdots, a_{\sigma(i)}]$$

Since A is $[\cdot, \cdot]$-closed the above computations show $pF = f_1 + Kq_+F$ and $pR = r_1 + \pi q_+ F$, ending the first part of the proof.

At this point, consider the F-coderivation $FR - QF : \overline{S(s^{-1}M \oplus A)} \to \overline{S\,\mathrm{Cyl}_i}$, following closely the proof of Theorem 4.1 in [5] (the arXiv version), we see that

$$p(FR - QF) = (f_1 + Kq_+F)R - qF = f_1 pR + Kq_+FR - (q_1 + q_+)F =$$
$$= f_1(r_1 + \pi q_+ F) + Kq_+FR - q_1(f_1 + Kq_+F) - q_+F =$$
$$= (f_1 r_1 - q_1 f_1) + (f_1 \pi - \mathrm{id}_{\mathrm{Cyl}_i} - q_1 K) q_+ F + Kq_+FR =$$
$$= (f_1 r_1 - q_1 f_1) + Kq_1 q_+ F + Kq_+FR$$

By the yet to be proved claim

$$q_1{}^2 = pQ^2 = q_1 Q + q_+ Q = q_1{}^2 + q_1 q_+ + q_+ Q \Rightarrow q_1 q_+ = -q_+ Q$$

and it follows

$$p(FR-QF)=(f_1r_1-q_1f_1)+Kq_+(FR-QF) \tag{4.2}$$

We put $FQ-RF=H=(h_1,\ldots,h_i,\ldots)$, so that (4.2) reads $pH=h_1+Kq_+H$. A plain direct inspection shows that $h_1(s^{-1}m)=0$, $h_1(a)=(s^{-1}P^{\perp}D^2a,0,0)$: using this as inductive basis, we show that H is given by

$$h_i(a_1\odot\cdots\odot a_i)=\left(s^{-1}\frac{1}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)P^{\perp}[\cdots[D^2a_{\sigma(1)},a_{\sigma(2)}]\cdots,a_{\sigma(i)}],0,0\right)$$

and $H=0$ elsewhere. This follows by induction, using (4.2) and

$$H^i_j(a_1\odot\cdots\odot a_i)=\sum_{k=1}^{i-j+1}\sum_{\sigma\in S(k,i-k)}\varepsilon(\sigma)h_k(a_{\sigma(1)}\odot\cdots\odot a_{\sigma(k)})\odot F^{j-1}_{i-k}(a_{\sigma(k+1)}\odot\cdots\odot a_{\sigma(i)})$$

together with a series of computations completely similar to the previous ones, that in the end lead to

$$\begin{aligned}p_Mq_+H(a_1\odot\cdots\odot a_i)&=\frac{1}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)[\cdots[D^2a_{\sigma(1)},a_{\sigma(2)}]\cdots,a_{\sigma(i)}]+\\&+\sum_{k=1}^{i-1}\frac{B_{i-k}}{k!(i-k)!}\sum_{\sigma\in S_i}\varepsilon(\sigma)\overbrace{[\cdots[}^{i-k}P([\cdots[D^2a_{\sigma(1)},a_{\sigma(2)}]\cdots]),a_{\sigma(k+1)}]\cdots,a_{\sigma(i)}]\end{aligned} \tag{4.3}$$

and $p_Mq_+H=0$ elsewhere: composing with K proves the inductive step.

In order to conclude the argument, first notice that ${r^2}_1={r_1}^2$ is clearly as given in the claim of the proposition (cf. Equation (4.1)), then we follow again the proof of [5], Theorem 4.1, to get

$$\begin{aligned}pR^2=(r_1+\pi q_+F)R={r_1}^2+r_1\pi q_+F+\pi q_+FR=\\={r^2}_1+r_1\pi q_+F+\pi q_+QF+\pi q_+(FR-QF)=\\={r^2}_1+(r_1\pi-\pi q_1)q_+F+\pi q_+H={r^2}_1+\pi q_+H\end{aligned}$$

Finally (4.3) imply the thesis about R^2, and the last statement is obvious from the given proof.

It remains to prove the claim that $Q^2={q_1}^2$, for this let $H_i=M\times^h_M L$ be the homotopy fiber product along id_M and i, let D act in an obvious way as a derivation of H_i and let $(s^{-1}H_i,S)$ be the Quillen construction on $(H_i,D,[\cdot,\cdot])$: since D is a derivation, in this case it is plain that $S^2={s_1}^2$. We can define homotopy retraction data from $(s^{-1}H_i,s_1)$ to (Cyl_i,q_1) as in Theorem 3.3 and Q is induced by S this way: noticing that neither q_+, s_+ nor the homotopy retraction data depend on D, this follows for instance from the same statement for $D=0$, thus it follows already from Theorem 3.3. Finally if $F:\overline{S\,\mathrm{Cyl}_i}\to\overline{S(s^{-1}H_i)}$ is such that $pF=f_1+Ks_+F$, reasoning as before we see $p(FQ-SF)=(f_1q_1-s_1f_1)+Ks_+(FQ-SF)$ and $pQ^2={q_1}^2+\pi s_+(FQ-SF)$. Starting with $(f_1q_1-s_1f_1)=0$ the first identity implies inductively that $FQ-SF=0$, and then the second implies the claim. □

Remark 4.5. When $L\subset M$ is an ideal, then $P:M\to A$ is a morphism of graded Lie algebras. One can check that in this case $\Phi(m)_i(a_1\odot\cdots\odot a_i)=\frac{B_i(1)}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)[\cdots[Pm,a_{\sigma(1)}]\cdots,a_{\sigma(i)}]$, while $\Phi(D)_1=PD$ and $\Phi(D)_i=0$ for $i\geq 2$. The resulting $L_\infty[1]$-structure on $s^{-1}M\oplus A$ is then essentially the same as the one defined in Theorem 3.2, that is the mapping cocone of the morphism P. Notice the case $L=0$.

The previous proposition already contains the necessary ingredients to the proof of Theorem 1.2, to which we now proceed.

Proof. (of Theorem 1.2) This is similar (tough in a certain sense it goes backwards) to the proof of Theorem 2 in [16]. We expand the identity

$$0 = r^2{}_{i+2}(s^{-1}m \odot s^{-1}n \otimes a_1 \odot \cdots \odot a_i) = r_{i+1}(r_2(s^{-1}m \odot s^{-1}n) \otimes a_1 \odot \cdots \odot a_i) +$$

$$+(-1)^{|m|+1} \sum_{k=0}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) r_{i-k+2}(s^{-1}m \otimes r_{k+1}(s^{-1}n \otimes a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) +$$

$$+(-1)^{|m|(|n|+1)} \sum_{k=0}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) r_{i-k+2}(s^{-1}n \otimes r_{k+1}(s^{-1}m \otimes a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) =$$

$$= (-1)^{|m|} \Phi([m,n])_i(a_1 \odot \cdots \odot a_i) +$$

$$(-1)^{|m|+1} \sum_{k=0}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) \Phi(m)_{i-k+1}(\Phi(n)_k(a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) +$$

$$(-1)^{|m|+1}(-1)^{|m||n|+1} \sum_{k=0}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) \Phi(n)_{i-k+1}(\Phi(m)_k(a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) =$$

$$= (-1)^{|m|} \Phi([m,n])_i(a_1 \odot \cdots \odot a_i) - (-1)^{|m|} [\Phi(m), \Phi(n)]_i(a_1 \odot \cdots \odot a_i)$$

to deduce Equation (1.1).

Similarly we expand the identity

$$0 = r^2{}_{i+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_i) = -r_{i+1}(s^{-1}Dm \otimes a_1 \odot \cdots \odot a_i) + r_{i+1}(Pm \odot a_1 \odot \cdots \odot a_i) +$$

$$+ \sum_{k=1}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) r_{i-k+1}(r_{k+1}(s^{-1}m \otimes a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) +$$

$$+(-1)^{|m|+1} \sum_{k=1}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) r_{i-k+2}(s^{-1}m \otimes r_k(a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) =$$

$$= -\Phi(Dm)_i(a_1 \odot \cdots \odot a_i) +$$

$$+ \sum_{k=0}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) \Phi(D)_{i-k+1}(\Phi(m)_k(a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) -$$

$$-(-1)^{|m|} \sum_{k=1}^{i} \sum_{\sigma \in S(k,i-k)} \varepsilon(\sigma) \Phi(m)_{i-k+1}(\Phi(D)_k(a_{\sigma(1)} \odot \cdots \odot a_{\sigma(k)}) \odot a_{\sigma(k+1)} \odot \cdots \odot a_{\sigma(i)}) =$$

$$= -\Phi(Dm)_i(a_1 \odot \cdots \odot a_i) + [\Phi(D), \Phi(m)]_i(a_1 \odot \cdots \odot a_i)$$

to prove Equation (1.3) when $|D| = 1$.

Finally

$$\frac{1}{2}\Phi([D,D])_i(a_1 \odot \cdots \odot a_i) = r^2{}_i(a_1 \odot \cdots \odot a_i) = \frac{1}{2}[\Phi(D), \Phi(D)]_i(a_1 \odot \cdots \odot a_i)$$

and by polarization we get Equation (1.2) when both D_1, D_2 are degree one.

To conclude we proceed as in the proof of [16] Theorem 3. We observe that all the constructions made so far would continue to make sense if we were working with a general commutative DGA Λ (over $\mathbb{K}$) as the base ring; though is not clear to the author whether some mild assumption should be necessary for the proof of Proposition 4.4 to run smoothly, this is certainly the case if Λ is a free graded commutative $\mathbb{K}$-algebra and M, L and A are free Λ-modules.

For Λ a free graded commutative $\mathbb{K}$-algebra, we repeat the construction of Definition 4.1 to the split Lie Λ-algebra $\Lambda\otimes M=(\Lambda\otimes L)\oplus(\Lambda\otimes A)$: working everywhere over Λ we get a correspondence $\Phi_\Lambda:\operatorname{Der}_\Lambda(\Lambda\otimes M,\Lambda\otimes L)\to\operatorname{Coder}_\Lambda(\overline{S_\Lambda(\Lambda\otimes A)})$, by the above this preserves the Lie bracket of degree one elements. There are isomorphisms $\Lambda\otimes\operatorname{Der}(M,L)\cong\operatorname{Der}_\Lambda(\Lambda\otimes M,\Lambda\otimes L)$ and $\Lambda\otimes\operatorname{Coder}(\overline{SA})\cong\operatorname{Coder}_\Lambda(\overline{S_\Lambda(\Lambda\otimes A)})$, the second one sends $\lambda\otimes Q$ to the Λ-coderivation given in Taylor coefficients $((\lambda\otimes q)_1,\dots,(\lambda\otimes q)_i,\dots)$ by the rule $(\lambda\otimes q)_i((\lambda_1\otimes a_1)\odot\cdots\odot(\lambda_i\otimes a_i))=\pm_K\lambda\lambda_1\cdots\lambda_i\otimes q_i(a_1\odot\cdots\odot a_i)$, with $\pm_K$ the correct Koszul sign. Finally it is plain to see that under the given isomorphisms $\Phi_\Lambda=\mathrm{id}_\Lambda\otimes\Phi$ (i.e., higher derived brackets commute with scalar extension).

From the previous considerations (1.2) follows in general: we take $\Lambda=\Lambda(\lambda_1,\lambda_2)$ the free algebra over independent variables of degree $|\lambda_i|=1-|D_i|$, $i=1,2$, then

$$(-1)^{|D_1||\lambda_2|}\lambda_1\lambda_2\otimes\Phi([D_1,D_2])=\Phi_\Lambda([\lambda_1\otimes D_1,\lambda_2\otimes D_2])=$$
$$=[\Phi_\Lambda(\lambda_1\otimes D_1),\Phi_\Lambda(\lambda_2\otimes D_2)]=(-1)^{|D_1||\lambda_2|}\lambda_1\lambda_2\otimes[\Phi(D_1),\Phi(D_2)]$$

and factoring away $(-1)^{|D_1||\lambda_2|}\lambda_1\lambda_2$ implies (1.2).

Finally, to prove Equation (1.3) in general, we argue as before in order to conclude that the correspondence $\Phi:\operatorname{Der}(M,L)\ltimes M\to\operatorname{Coder}(SA):(D,m)\to\Phi(D)+\Phi(m)$ is a morphism of graded Lie algebras, where $\operatorname{Der}(M,L)\ltimes M$ is the semidirect product. □

We end the section with a list of corollaries to the given proof of Proposition 4.4.

Corollary 4.6. *Let $D\in\operatorname{Der}^1(M,L)$ such that $D^2=0$: then $\Phi(D)$ is a $L_\infty[1]$-structure on A. Let K_i be the homotopy fiber of the inclusion of DGLAs $i:(L,D,[\cdot,\cdot])\longrightarrow(M,D,[\cdot,\cdot])$, then the $L_\infty[1]$-algebra $(A,\Phi(D))$ is homotopy equivalent to the Quillen construction on K_i.*

Proof. The first claim is clear from Equation (1.2). For the second, notice that the mapping cocone $\mathrm{C}_i=s^{-1}L\oplus M$ of the inclusion i, with the $L_\infty[1]$-structure S defined in Theorem 3.2, sits inside the mapping cocylinder (Cyl_i,Q) as a $L_\infty[1]$-subalgebra, in fact a $L_\infty[1]$-ideal; similarly A with the $L_\infty[1]$-structure $\Phi(D)$ sits inside $(s^{-1}M\oplus A,R)$ of Proposition 4.4 as a $L_\infty[1]$-subalgebra (again, it is a $L_\infty[1]$-ideal). Also notice that the homotopy retraction data from Proposition 4.4 restrict to homotopy retraction data from (C_i,s_1) to $(A,\Phi(D)_1)$, then the proof of 4.4 shows that $\Phi(D)$ is the $L_\infty[1]$-structure on A induced by S via homotopical transfer along this homotopy retraction data. Finally, since (C_i,S) is a model for the Quillen construction on K_i the same is true for the homotopically equivalent $L_\infty[1]$-algebra $(A,\Phi(D))$. □

Remark 4.7. In the above hypotheses the computations in Proposition 4.4 easily imply that $(f_1,\dots,f_i,\dots)=F:\overline{SA}\to\overline{S(s^{-1}L)}$ defined by

$$f_i(a_1\odot\cdots\odot a_i)=s^{-1}\left(\frac{1}{i!}\sum_{\sigma\in S_i}\varepsilon(\sigma)P^\perp[\cdots[Da_{\sigma(1)},a_{\sigma(2)}]\cdots,a_{\sigma(i)}]\right)$$

for $i\geq1$ is a $L_\infty[1]$-morphism from $(A,\Phi(D))$ to the Quillen construction on $(L,D,[\cdot,\cdot])$.

Remark 4.8. It seems worthwhile to mention, in light of possible applications to deformation theory, that $a\in A^0$ is a solution to the Maurer-Cartan equation

$$\sum_{i\geq1}\frac{\Phi(D)_i(a^{\odot i})}{i!}=0$$

in the $L_\infty[1]$-algebra $(A,\Phi(D))$, if and only if $e^{-a}*0\in L^1$, where $\cdot*\cdot:\exp(M^0)\times M^1\to M^1$ denotes the Gauge action in the DGLA $(M,D,[\cdot,\cdot])$ (cf. [13]). This is not hard and left to the interested reader.

Remark 4.9. Suppose A is a generic algebraic complementary to L in M and $D \in \mathrm{Der}^1(M, L)$ satisfies $D^2 = 0$. As remarked in the introduction, putting $\Phi(D)_1(a) = PDa$, the homotopy retraction data defined in Proposition 4.4 still restrict to homotopy retraction data from the mapping cocone (C_i, S) to $(A, \Phi(D)_1)$, so we could use homotopical transfer of structure to define (generalized) higher derived brackets on A associated to D in this case as well.

Another corollary is an easy derivation of the following result, essentially proved in [3].

Corollary 4.10. *Suppose the graded Lie algebra M splits as $M = L \oplus A_1$, $M = L \oplus A_2$, with L, A_1, A_2 graded Lie subalgebras of M, and let $D \in \mathrm{Der}^1(M, L)$ such that $D^2 = 0$. Denote with $\Phi_k : \mathrm{Der}(M, L) \to \mathrm{Coder}(\overline{SA_k})$, $k = 1, 2$, the respective higher derived brackets constructions, then the $L_\infty[1]$-algebras $(A_1, \Phi_1(D))$, $(A_2, \Phi_2(D))$ from Corollary 4.6 are isomorphic.*

Proof. Let $k = 1, 2$, denote with $P_k : M \to A_k$ the projection with kernel L, with (C_i, S) the mapping cocone of the inclusion i and with $F_k : (A_k, \Phi_k(D)) \to (\mathrm{C}_i, S)$ the restriction of the $L_\infty[1]$-morphism constructed in the proof of Proposition 4.4 (cf. Corollary 4.6). In general, given a $L_\infty[1]$-morphism F with linear part an injective quasi-isomorphism f_1, then any DG-left inverse g_1 to f_1 can be lifted to a left inverse G to F (though we are missing a reference for this, it follows from general theory of $L_\infty[1]$-algebras as in [9]): applying this to the $L_\infty[1]$-morphism F_k on the one hand, and to the DG-morphism $\pi_k : (\mathrm{C}_i, s_1) \to (A_k, \Phi_k(D)_1) : (s^{-1}l, m) \to P_k m$ on the other, we get a $L_\infty[1]$-morphism $G_k : (\mathrm{C}_i, S) \to (A_k, \Phi_k(D))$, left inverse to F_k and with linear part π_k. Finally, the $L_\infty[1]$-morphism $G_2F_1 : (A_1, \Phi_1(D)) \to (A_2, \Phi_2(D))$ is an isomorphism, since its linear part is the DG-isomorphism $(A_1, P_1D) \to (A_2, P_2D) : a \to P_2a$ (notice $P_2 - P_2P_1 = P_2P_1^\perp = 0$, since $P_1^\perp(M) = L$). □

Remark 4.11. Let $i : (L, D, [\cdot,\cdot]) \to (M, D, [\cdot,\cdot])$ be an inclusion of DGLAs, and let (N, d) be the quotient DG space. To give an algebraic complementary A to L in M is the same as to give a section $\sigma : N \to M$, splitting the exact sequence of graded spaces

$$0 \to L \to M \to N \to 0.$$

Using homotopical transfer from the mapping cocone (C_i, R), as in Remark 4.9, any such a section determines a $L_\infty[1]$-structure on N, with linear part d. Then, with the same proof as in the previous corollary, we see that different choices of σ lead to isomorphic $L_\infty[1]$-structures on (N, d).

Corollary 4.12. *In the hypotheses of Corollary 4.6, let (N, d) be the DG quotient space of the inclusion $(L, D) \to (M, D)$. If the exact sequence*

$$0 \to (L, D) \to (M, D) \xrightarrow{p} (N, d) \to 0$$

splits in the category of DG spaces, i.e., if there exists a DG morphism $s : (N, d) \to (M, D)$ such that $ps = \mathrm{id}_N$, then the $L_\infty[1]$-algebra $(A, \Phi(D))$ is homotopy abelian.

Proof. In the given hypotheses there exist a DG complementary A' to L in M, i.e., $M = L \oplus A'$ and $D(A') \subset A'$. Let $P' : M \to A'$ be the projection with kernel L, and define homotopy retraction data from the mapping cocone (C_i, S) to $(A', P'D)$ as in the proof of Proposition 4.4 (cf. Corollary 4.6 and Remark 4.9): as observed in the previous remark, the thesis will follow as in the proof of Corollary 4.10 if we prove that the $L_\infty[1]$-structure on A' induced by homotopical transfer from (C_i, S) is the abelian one given by $P'D$.

Notice that since A' is D-closed, f_1 is given by $f_1 : A' \to \mathrm{C}_i : a \to (0, a)$. Consider f_1 as a linear morphism of coalgebras: since it is plain to see that $s_+f_1 = 0$ it follows that $F = f_1$ is the

unique solution to $pF = f_1 + Ks_+F$, but then we also see that the transferred $L_\infty[1]$-structure on A' is the one with corestriction

$$P'D + \pi s_+ F = P'D + \pi s_+ f_1 = P'D$$

□

5. Examples and applications

Example 5.1. Recall (cf. [15, 6]) that every $L_\infty[1]$-structure can be obtained via the higher derived brackets construction. Let V be a graded space and identify it with an abelian subalgebra of $\mathrm{Coder}(SV)$ via the embedding $v \to L_v$ from Remark 2.2, splitting the exact sequence

$$0 \to \mathrm{Coder}(\overline{SV}) \xrightarrow{i} \mathrm{Coder}(SV) \xrightarrow{e} V \to 0$$

from Remark 2.1. The higher derived brackets construction define a morphism of graded Lie algebras $\Phi : \mathrm{Coder}(SV) \to \mathrm{Coder}(SV) : R \to \Phi(R)$, we claim that this is just the identity.

In fact, since V is abelian and $P : \mathrm{Coder}(SV) \to V : R \to R(1)$, we see that $\Phi(R)$ is given in Taylor coefficients by $\Phi(R)_0 = R(1) = r_0$ and $\Phi(R)_n(v_1 \odot \cdots \odot v_n) = [\cdots[R, L_{v_1}]\cdots, L_{v_n}](1) = r_n(v_1 \odot \cdots \odot v_n)$ (cf. Remark 2.2), i.e., $\Phi(R) = R$ as claimed. If Q is a $L_\infty[1]$-structure on V, i.e., $Q \in \mathrm{Coder}^1(\overline{SV})$ and $Q^2 = 0$, by the above $Q = \Phi(Q) = \Phi(\mathrm{ad}\, Q)$ (cf. Remark 4.2): then Corollary 4.6 implies the following result, already obtained in [4].

Theorem 5.2. *Every $L_\infty[1]$-algebra (V, Q) is homotopy equivalent to the Quillen construction on the homotopy fiber of the inclusion of DGLAs*

$$i : (\mathrm{Coder}(\overline{SV}), [Q, \cdot], [\cdot, \cdot]) \to (\mathrm{Coder}(SV), [Q, \cdot], [\cdot, \cdot])$$

Remark 5.3. An explicit homotopy equivalence $F = (f_1, \ldots, f_i, \ldots)$ from (V, Q) to the Quillen construction on the above homotopy fiber can be determined by the computations in this paper and those in [5]: first from Remark 4.7 we get a $L_\infty[1]$-morphism $\mathrm{Ad}_\infty = (\mathrm{Ad}_1, \ldots, \mathrm{Ad}_i, \ldots)$ from (V, Q) to the Quillen construction on $(\mathrm{Coder}(\overline{SV}), [Q, \cdot], [\cdot, \cdot])$, given explicitly for $i \geq 1$ by

$$\mathrm{Ad}_i(v_1 \odot \cdots \odot v_i) = s^{-1}([\cdots[Q, L_{v_1}]\cdots, L_{v_i}] - L_{q_i(v_1 \odot \cdots \odot v_i)})$$

(i.e., $s\,\mathrm{Ad}_i(v_1 \odot \cdots \odot v_i)$ is given in Taylor coefficients by $s\,\mathrm{Ad}_i(v_1 \odot \cdots \odot v_i)_k(v_{i+1} \odot \cdots \odot v_{i+k}) = q_{i+k}(v_1 \odot \cdots \odot v_{i+k})$ for $k \geq 1$). This is the $L_\infty[1]$-generalization of the adjoint morphism of a DGLA introduced by Chuang and Lazarev in [4]. Then the homotopy equivalence F is explicitly given by

$$f_1(v) = s^{-1}\left(s\,\mathrm{Ad}_1(v), t \cdot s\,\mathrm{Ad}_1(v) + dt \cdot L_v\right)$$

$$f_i(v_1 \odot \cdots \odot v_i) = s^{-1}\left(s\,\mathrm{Ad}_i(v_1 \odot \cdots \odot v_i), t^i \cdot s\,\mathrm{Ad}_i(v_1 \odot \cdots \odot v_i) + (t^i - t) \cdot L_{q_i(v_1 \odot \cdots \odot v_i)}\right)$$

Notice that this construction offer a way to rectify a given $L_\infty[1]$-algebra (i.e., to find an homotopy equivalent DGLA) alternative to Quillen's L functor (cf. [8]); also notice, however, that we can not rectify morphisms of DGLAs this way.

As a consequence of Corollary 4.12 we also obtain the following result.

Theorem 5.4. *Let (V, Q) be a $L_\infty[1]$-algebra. If the evaluation morphism*

$$e : (\mathrm{Coder}(SV), [Q, \cdot]) \to (V, q_1) : R \to R(1)\,(= r_0\,)$$

admits a DG right inverse, then the $L_\infty[1]$-algebra (V, Q) is homotopy abelian.

Remark 5.5. The converse of the above theorem is also true , so the hypothesis is a necessary and sufficient condition for a $L_\infty[1]$-algebra (V,Q) to be homotopy abelian. This can be reformulated as the vanishing of the map induced by the adjoint $H(\mathrm{Ad}_1): H(V,q_1)\to H(\mathrm{Coder}(\overline{SV}),[Q,\cdot])[1]$ from the tangent cohomology to the reduced Chevalley-Eilenberg cohomology of (V,Q), with the $L_\infty[1]$-adjoint morphism Ad_∞ defined as in the previous remark.

Example 5.6. The higher derived brackets construction has been applied to the study of coisotropic deformations, cf. [14, 3, 6]. Let X be a differentiable manifold, TX the tangent bundle, we denote with $\mathcal{V}^*_X=\Gamma(\bigwedge^* TX)$ the Gerstenhaber algebra of multivector fields on X, equipped with the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}:\mathcal{V}^i_X\otimes\mathcal{V}^j_X\to\mathcal{V}^{i+j-1}_X$. Recall that a Poisson structure on X is the datum of a bivector $\pi\in\mathcal{V}^2_X$ such that $[\pi,\pi]_{SN}=0$: as well known (cf. for instance [1], Section 3) this induces a Poisson bracket $\{\cdot,\cdot\}_\pi$ on the algebra $C(X)$ of smooth functions on X, a DGLA structure $(s^{-1}\mathcal{V}^*_X,[\pi,\cdot]_{SN},[\cdot,\cdot]_{SN})$ on the desuspension $s^{-1}\mathcal{V}^*_X$, and finally an anchor map $\pi^\#:T^*X\to TX$ given by contraction with π.

Let $Z\subset X$ be a closed smooth submanifold, recall that Z is coisotropic if the vanishing ideal $I(Z)\subset C(X)$ is $\{\cdot,\cdot\}_\pi$-closed, equivalently, if $\pi^\#(N^*Z)\subset TZ$, where $N^*Z\subset T^*X$ is the annihilator of TZ. Let NZ be the normal bundle of Z in X, $\mathcal{N}^*_Z=\Gamma(\bigwedge^* NZ)$: restriction to Z followed by projection induces an algebra epimorphism $\mathcal{V}^*_X\to\mathcal{N}^*_Z$, let $\mathcal{L}^*_Z\subset\mathcal{V}^*_X$ be defined by the exact sequence

$$0\to\mathcal{L}^*_Z\to\mathcal{V}^*_X\to\mathcal{N}^*_Z\to 0$$

As in [1], Proposition 5.2, $\mathcal{L}^*_Z$ is a sub Gerstenhaber algebra of $\mathcal{V}^*_X$, and Z is coisotropic if and only if $\pi\in\mathcal{L}^2_Z$. When X is the total space of a vector bundle on Z (and Z is embedded as the zero section) the above sequence admits a natural splitting, whose desuspension sends $s^{-1}\mathcal{N}^*_Z$ onto an abelian subalgebra of $s^{-1}\mathcal{V}^*_X$, cf. for instance [3, 14]: in general one reduces to this situation with the choice of an embedding of NZ onto a tubular neighborhood of Z in X. For a coisotropic Z the higher derived brackets $\Phi(\pi)=\Phi(\mathrm{ad}\,\pi)$ (cf. Remark 4.2) induce a $L_\infty[1]$-structure on $s^{-1}\mathcal{N}^*_Z$, moreover it follows from Corollary 4.10 (cf. also Remark 4.11) that the resulting $L_\infty[1]$-algebra $(s^{-1}\mathcal{N}^*_Z,\Phi(\pi))$ is independent from the involved choice up to isomorphism, obtaining a result already proved in [3].

It is known (cf. [14]) that the $L_\infty[1]$-algebra $(s^{-1}\mathcal{N}^*_X,\Phi(\pi))$ governs the functor of infinitesimal coisotropic deformations of Z in X (via the associated deformation functor, cf. [13, 5]): as homotopically equivalent $L_\infty[1]$-algebras determine the same deformation functor, Corollary 4.6 implies the following theorem.

Theorem 5.7. *Let (X,π) be a (differentiable) Poisson manifold, $Z\subset X$ a coisotropic submanifold, then the homotopy fiber K_i of the inclusion of DGLAs*

$$i:(s^{-1}\mathcal{L}^*_Z,[\pi,\cdot]_{SN},[\cdot,\cdot]_{SN})\to(s^{-1}\mathcal{V}^*_X,[\pi,\cdot]_{SN},[\cdot,\cdot]_{SN})$$

governs the functor of infinitesimal coisotropic deformations of Z in X.

The same result could have been proved by the methods of [1].

Example 5.8. Let A be a unital graded algebra, denote with A_L the corresponding graded Lie algebra, with the commutator bracket, and with A_J the corresponding Jordan algebra, with Jordan product $a\circ b=\frac{1}{2}(ab+(-1)^{|a||b|}ba)$. Identify A_L with a Lie subalgebra of $\mathrm{Hom}(A)$ via the embedding $\lambda: A_L\to\mathrm{Hom}(A): a\to\lambda(a)$, where $\lambda(a)$ is the operator of left multiplication by a. We have a projection $P:\mathrm{Hom}(A)\to A_L: f\to\lambda(f(1))$, whose kernel $L=\{f\in\mathrm{Hom}(A)\text{ s.t. }f(1)=0\}$ is a graded Lie subalgebra of $\mathrm{Hom}(A)$. We are in the set up of Section 4, so higher derived brackets define a morphism of graded Lie algebras $\Phi:\mathrm{Hom}(A)\to\mathrm{Coder}(SA): f\to\Phi(f)$. When A is (graded) commutative, the $\Phi(f)_i$ defined in this way are the usual higher Koszul brackets

associated to f [10, 15], in the non commutative case we expect to recover the hierarchy of non-abelian higher Koszul brackets introduced by Bering in [2], Section 3 (though we omitted to verify this passage).

To get a taste of how do the just defined brackets look like in the non commutative case, one can easily verify that $\Phi(f)_1(a) = f(a) - \frac{1}{2}(f(1)a + (-1)^{|a||f|}af(1)) = f(a) - f(1)\circ a$ for every $a\in A$; as another example let $f\in\operatorname{Hom}(A)$ such that $f(1)=0$, then it is plain to check that $\Phi(f)_2(a\odot b) = f(a\circ b) - f(a)\circ b - (-1)^{|a||f|}a\circ f(b)$. We could define a family of subspaces $D_k = \{f\in\operatorname{Hom}(A)\text{ s.t. }\Phi(f)_i = 0,\ \forall\, i>k\}$ for $k\geq 0$, since Φ is a Lie algebras morphism the identity $[D_i, D_j]\subset D_{i+j-1}$ follows immediately, in particular $\bigcup_{k\geq 0}D_k$ is a graded Lie subalgebra of $\operatorname{Hom}(A)$: when A is commutative $D_k\subset\operatorname{Hom}(A)$ is the subspace of differential operators of order k on A, it is not clear to the author whether the D_k define interesting classes of operators in the non commutative case as well (cf. also the next remark).

Remark 5.9. We believe that meaningful constructions of "higher derived operations" could be performed in different operadic contexts by the methods of this paper. For instance let B be a graded algebra that splits, as a graded space, in the direct sum $B = A\oplus C$ of graded subalgebras A and C, then for every $D\in\operatorname{Der}^1(B,A)$ such that $D^2=0$ it is induced via homotopical transfer of structure an $A_\infty[1]$-structure on C, that is a squaring to zero degree one coderivation on the reduced tensor coalgebra $\overline{TC}$ over C. We believe, and some sketchy computations seem to confirm it, that this correspondence extends to a morphism of graded Lie algebras $\Phi:\operatorname{Der}(B,A)\to\operatorname{Coder}(\overline{TC})$. Moreover we expect this to commute with the constructions of this paper in the following sense: if we denote with A_L and B_L the corresponding graded Lie algebras, with bracket the commutator, and with $\operatorname{Sym}:\operatorname{Coder}(\overline{TC})\to\operatorname{Coder}(\overline{SC})$ the symmetrization operator, then the following should be a commutative diagram of morphisms of graded Lie algebras

$$\begin{tikzcd} \operatorname{Der}(B,A) \arrow[r, "\Phi"] \arrow[d, hook] & \operatorname{Coder}(\overline{TC}) \arrow[d, "\operatorname{Sym}"] \\ \operatorname{Der}(B_L,A_L) \arrow[r, "\Phi"] & \operatorname{Coder}(\overline{SC}) \end{tikzcd}$$

where the morphism in the bottom line is the one from Theorem 1.2: there may be a connection with some of the results of [11]. In the set up of the previous Example, one could perhaps give a more interesting construction of nonsymmetric higher Koszul brackets working along these lines.

Example 5.10. Let $(L,D,[\cdot,\cdot])$ be a DGLA, then $L = L^{\geq 0}\oplus L^{<0}$, where $L^{\geq 0} = \oplus_{i\geq 0}L^i$ (resp.: $L^{<0} = \oplus_{i<0}L^i$): notice that both $L^{\geq 0}$ and $L^{<0}$ are Lie subalgebras and $D(L^{\geq 0})\subset L^{\geq 0}$, thus we are in the set up of Section 4 and via the higher derived bracket construction we obtain a $L_\infty[1]$-structure $\Phi(D)$ on $L^{<0}$. To compute this explicitly, it is convenient to rewrite the brackets in Definition 4.1 as

$$\Phi(D)_i(l_1\odot\cdots\odot l_i) = \sum_{k=2}^{i}\frac{B_{i-k}}{k!(i-k)!}\sum_{\sigma\in S_i}\varepsilon(\sigma)\overbrace{[\cdots[}^{i-k}P([\cdots[P^\perp Dl_{\sigma(1)}, l_{\sigma(2)}]\cdots]), l_{\sigma(k+1)}]\cdots, l_{\sigma(i)}]$$

for $i\geq 3$, $\Phi(D)_2(l_1\odot l_2) = -B_1\sum_{\sigma\in S_2}\varepsilon(\sigma)P[P^\perp Dl_{\sigma(1)}, l_{\sigma(2)}]$. The reader should have no problem in recognizing the equivalence of the above expression and the one in Definition 4.1, using the fact that A is $[\cdot,\cdot]$-closed and the identity $\sum_{k=0}^{i-1}B_k\begin{pmatrix} i\\ k\end{pmatrix} = 0$ for $i\geq 2$. Notice that $P^\perp D$ acts on $L^{<0}$ as D on L^{-1} and 0 elsewhere, also notice that all the P in the above formula are irrelevant,

since they apply to elements already in $L^{<0}$. Thus

$$\Phi(D)_i(l_1 \odot \cdots \odot l_i) = \left(\sum_{k=2}^{i} \frac{B_{i-k}}{k!(i-k)!}\right) \sum_{\sigma\in S_i} \varepsilon(\sigma)[\cdots[P^{\perp}Dl_{\sigma(1)}, l_{\sigma(2)}]\cdots, l_{\sigma(i)}] =$$

$$= -\frac{B_{i-1}}{(i-1)!} \sum_{\sigma\in S_i} \varepsilon(\sigma)[\cdots[P^{\perp}Dl_{\sigma(1)}, l_{\sigma(2)}]\cdots, l_{\sigma(i)}]$$

for $i \geq 3$, and the same formula applies for $i = 2$; finally $\Phi(D)_1 = PD$ which is 0 on L^{-1} and D on $L^{<-1}$. These are essentially the same as the ones introduced by Getzler in [7].

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